APPENDIX I: FORMULATION OF A GENERIC ALGORITHM FOR EARTH LOCATING DATA FROM NOAA POLAR SATELLITES

This appendix is a version of a paper written by James K. Ellickson (NOAA/NESDIS), Marie D. Henry(NOAA/NESDIS), Dr. C.K. Wong and Dr. Om P. Sharma(both of Science and Systems Applications, Inc) in August 1988 and included in NOAA Technical Memorandum NESS 107, Revision 1, as Appendix C. Although it was written almost 20 years ago, the information contained herein is still pertinent to the new NOAA KLM series satellites. The present version has been modified by Irv Ruff (NOAA/NESDIS), to include provision for instrument mounting errors, and by Howard Carney (NOAA/NESDIS), to include provision for geodetic subpoint and to correct some inconsistencies.

I.1 GENERAL

This document describes a general algorithm for computing Earth location values for data from scanning radiometers flown on three-axis stabilized polar orbiting satellites. Given the following information about any data point, a corresponding latitude and longitude on Earth can by computed:

1) Position and velocity of the satellite as a function of time

$$\vec{P}_{sat}(t) = (X_{sat}, Y_{sat}, Z_{sat})_I$$

where $\vec{P}_{sat}(t)$ is the position vector of the satellite and its components are expressed in the earth-centered-inertial coordinate system I (e.g., equator and equinox of date) at a time in a standard time system (such as UTC).

$$\vec{V}_{sat}(t) = (\dot{X}_{sat}, \dot{Y}_{sat}, \dot{Z}_{sat})_I$$

where $\vec{V}_{sat}(t)$ is the velocity of the satellite relative to the same inertially fixed coordinate system I at time t with its components expressed in that system.

2) Rotation of the Earth in the same time system, t, relative to the earth-fixed-inertial coordinate system, I, with a rotation angle written G(t), commonly called the “Greenwich Hour Angle”.

3) Misalignments of the instrument from the coordinate system in which the nadir position of the scanner points towards the subsatellite point and scanning is perpendicular to the scan-axis vector/satellite subpoint vector plane. (see not below.)

a. the misalignments are constant instrument mounting and/or attitude errors

b. misalignments or attitude errors as functions of time in the t system

4) The time difference between the t system and the time-tagging system of the satellite, (i.e., the onboard computer).

5) The timing and angular displacements of each data sample in the scanning cycle.

NOTES ON MISALIGNMENTS OF THE INSTRUMENTS:

The constant instrument mounting and/or time dependent attitude errors for NOAA satellites are minimized by the Attitude Determination and Control System (ADACS) onboard the spacecraft. This system orients the satellite in such a way that the nadir position of the scanning instruments always points toward the geodetic satellite subpoint. The residual errors of this system are reported in the data stream and could be used to correct the earth locations calculations in the paper. However, the handling of these misalignment errors, as a function of time, is somewhat complicated and beyond the intended scope of this algorithm description. These misalignments are included here for the sake of completion.

I.2 CALCULATING THE EARTH COORDINATES OF A SCAN SPOT

Consider an inertial coordinate system, I, whose origin is the center of the Earth (Figure I.2-1). The line joining the center of the earth to the Vernal Equinox constitutes the X-axis. The Z-axis is perpendicular to the equatorial plane and in the direction of the North Pole. The Y-axis is defined such that the vectors $\vec{X}, \vec{Y}, \vec{Z}$ constitute a right handed coordinate system. Let $\vec{P}_{sat}$ and $\vec{P}_{spot}$ be position vectors of the satellite and the scan spot, respectively. Then the position of the scan spot on the earth in the inertial coordinate system can be expressed in equation I-1.

$$\begin{Vmatrix} X_{spot} \\ Y_{spot} \\ Z_{spot} \end{Vmatrix} = \begin{Vmatrix} X_{sat} \\ Y_{sat} \\ Z_{sat} \end{Vmatrix} + R \begin{Vmatrix} d_x \\ d_y \\ d_z \end{Vmatrix} \tag{I-1}$$

or

$$\vec{P}_{spot} = \vec{P}_{sat} + R\hat{d} \tag{I-2}$$

Figure I.2-1. The geometry of the satellite and its scan spot relative to the earth and the earth-centered-inertial coordinate system.

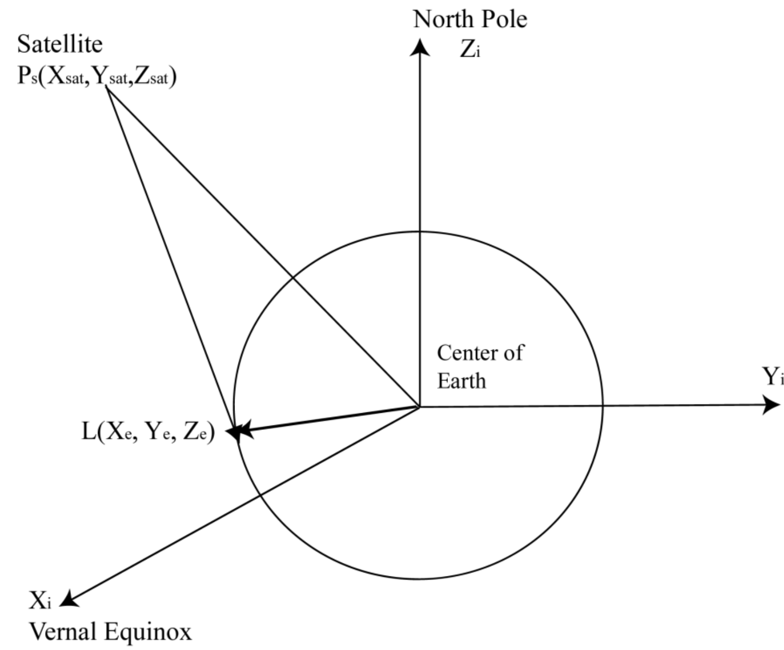

Figure I.2-1. The Geometry of the Satellite and Its Scan Spot Relative to the Earth and the Earth-centered-inertial Coordinate System

Where R is the range or distance from the satellite to the scan spot and (dx,dy,dz) are the direction cosines of the scan spot from the satellite. The subscript I designates the inertial coordinate system. See Figure I.2-1.

In order to solve for $P_{spot} = (X_{spot}, Y_{spot}, Z_{spot})_I$, a new coordinate system centered at the spacecraft is established; call it the nominal scanning coordinate system. The scanner mounting frame is taken as the origin. The positive X_{ns}-axis is in the direction of the satellite's subpoint (See I.3 "Defining the Satellite Subpoint"). The Z_{ns}-axis is along the nominal spin axis of the mirror, perpendicular to X_{ns} and positive in the direction of the velocity vector. The Y_{ns}-axis completes a right handed system. If there are no misalignments, the instrument mirror will scan perpendicular to the X_{ns}-Z_{ns} plane. See Figure I.2- 2.
Define

$$\hat{P} \equiv (X_p, Y_p, Z_p)_I \qquad \text{(I-3)}$$

where X_p, Y_p and Z_p are the direction cosines, in earth-centered-inertial coordinates, of the

satellite subpoint from the satellite.

The speed of the satellite relative to the earth-centered-inertial frame is

$$\left|\vec{V}_{sat}\right| = \sqrt{\dot{X}_{sat}^2 + \dot{Y}_{sat}^2 + \dot{Z}_{sat}^2}$$

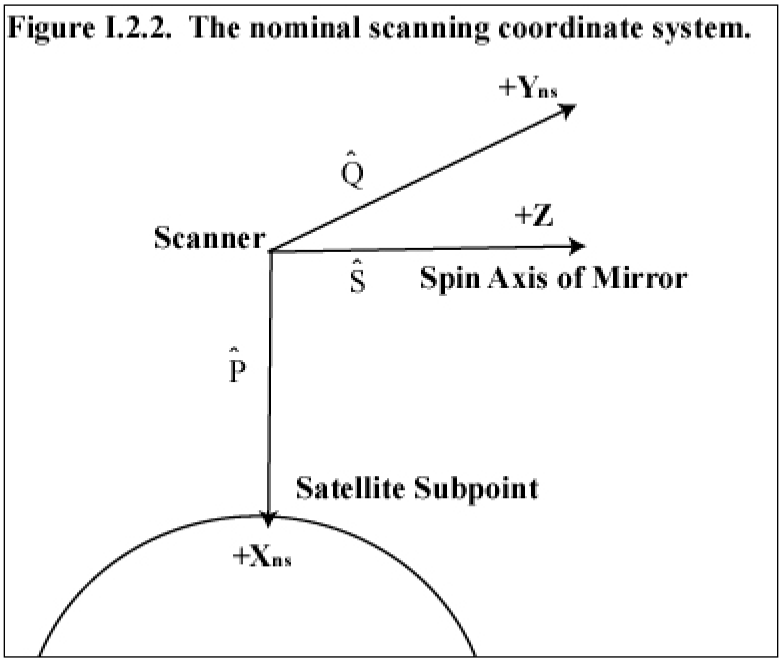

Figure I.2-2. The Nominal Scanning Coordinate System.

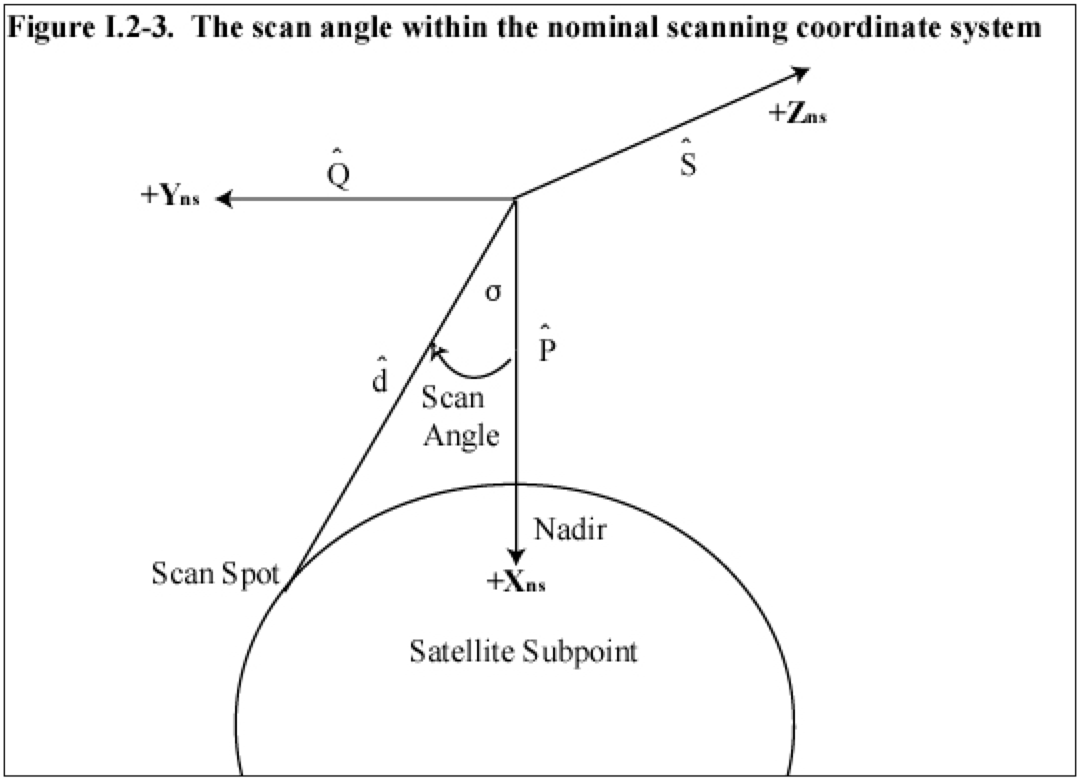

Figure I.2-3. The Scan Angle Within the Nominal Scanning Coordinate System.

Define

$$\hat{v} \equiv (\dot{X}_{sat} / |\vec{V}_{sat}|, \dot{Y}_{sat} / |\vec{V}_{sat}|, \dot{Z}_{sat} / |\vec{V}_{sat}|)_I$$

$$\equiv (X_v, Y_v, Z_v)_I \qquad \text{(I-4)}$$

where X_v, Y_v and Z_v are the direction cosines of the satellites velocity relative to the earth-centered-inertial frame and expressed as components in the earth-centered-inertial system.

$$\hat{Q} = \frac{\hat{v} \times \hat{P}}{|\hat{v} \times \hat{P}|} = \frac{(Y_v Z_p - Z_v Y_p, Z_v X_p - X_v Z_p, X_v Y_p - Y_v X_p)_I}{|\hat{v} \times \hat{P}|}$$

$$= (X_q, Y, Z_q)_I \qquad \text{(I-5)}$$

where

$$|\hat{v} \times \hat{P}| = \sqrt{(Y_v Z_p - Z_v Y_p)^2 + (Z_v X_p - X_v Z_p)^2 + (X_v Y_p - Y_v X_p)^2}$$

and X_q, Y_q and Z_q are the direction cosines of Q in the earth-centered-inertial system. The spin vector

$$\hat{S} = \hat{P} \times \hat{Q} = \frac{\hat{P} \times (\hat{v} \times \hat{P})}{\left|\hat{v} \times \hat{P}\right|}$$

$$= (Y_p Z_q - Z_p Y_q, Z_p X_q - X_p Z_q, X_p Y_q - Y_p X_q)_I$$

$$= (X_{spin}, Y_{spin}, Z_{spin})_I \tag{I-6}$$

where X_{spin}, Y_{spin}, and Z_{spin} are the direction cosines of $\hat{S}$ in the earth-centered-inertial system. In Figure I.2-3, the scanner direction $\hat{d}$ is measured in the PQS system, that is, the nominal scanning system. The unit vector $\hat{d}$ is really $\hat{P}$ rotated about $\hat{S}$ by the right-handed scan angle σ as shown in Figure I.2-3. It should be noted that the scan direction of the AVHRR instrument is right-handed and opposite that of the TOVS instruments. Each of the TOVS instruments scans in the same direction, sun to anti-sun, and its scan is a left-handed rotation about the scan axis.

To transfer any vector, $\vec{A}$, from the PQS coordinate system to the inertial system,

$$\vec{A}_I = \begin{Vmatrix} X_p & X_q & X_{spin} \\ Y_p & Y_q & Y_{spin} \\ Z_p & Z_q & Z_{spin} \end{Vmatrix} \vec{A}_{PQS} \tag{I-7}$$

Case 1: It there are no instrument mounting errors, the scan is measured in the nominal scanning coordinate system. The unit vector $\hat{d}$ (along the direction toward the scanned spot) in the PQS system can be written as:

$$\hat{d}_{PQS} = \begin{Vmatrix} \cos\sigma & -\sin\sigma & 0 \\ \sin\sigma & \cos\sigma & 0 \\ 0 & 0 & 1 \end{Vmatrix} \begin{Vmatrix} 1 \\ 0 \\ 0 \end{Vmatrix} \tag{I-8}$$

Using the coordinate transformation given by Eq. I.2-7 and employing Eqs I.2-3, I.2-5 and I.2-6, the vector can be written in the inertial system as:

$$\hat{d} = \begin{Vmatrix} X_p & X_q & X_{spin} \\ Y_p & Y_q & Y_{spin} \\ Z_p & Z_q & Z_{spin} \end{Vmatrix} \begin{Vmatrix} \cos\sigma & -\sin\sigma & 0 \\ \sin\sigma & \cos\sigma & 0 \\ 0 & 0 & 1 \end{Vmatrix} \begin{Vmatrix} 1 \\ 0 \\ 0 \end{Vmatrix} \tag{I-9}$$

Case 2: If there are instrument mounting errors, expressed as nonzero right-handed roll(R), pitch (P) and yaw (Y), then the unit vector in the PQS system in equation I-8 becomes:

$$\hat{d}_{PQS} = [B][C][D] \begin{Vmatrix} 1 \\ 0 \\ 0 \end{Vmatrix} \tag{I-10}$$

Where

$[B]$ is the rotation matrix about the yaw axis, nominally X_{ns}, for an angle (-Y) (to undo the yaw)

$$[B] = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & \cos Y & -\sin Y \\ 0 & \sin Y & \cos Y \end{Vmatrix} \tag{I-11}$$

$[C]$ is the rotation matrix about the pitch axis, nominally Y_{ns}, for an angle (-P) (to undo the pitch)

$$[C] = \begin{Vmatrix} \cos P & 0 & \sin P \\ 0 & 1 & 0 \\ -\sin P & 0 & \cos P \end{Vmatrix} \tag{I-12}$$

$[D]$ is the rotation matrix about the roll axis, nominally Z_{ns}, for an angle $-(\sigma+R)$ (to undo both the roll and the scan)

$$[D] = \begin{Vmatrix} \cos(\sigma + R) & -\sin(\sigma + R) & 0 \\ \sin(\sigma + R) & \cos(\sigma + R) & 0 \\ 0 & 0 & 1 \end{Vmatrix} \tag{I-13}$$

$$\hat{d} = \begin{Vmatrix} X_p & X_q & X_{spin} \\ Y_p & Y_q & Y_{spin} \\ Z_p & Z_q & Z_{spin} \end{Vmatrix} \hat{d}_{PQS} \tag{I-14}$$

Since $P_{spot} = P_{sat} + Rd$ and the point L lies on the surface of the Earth, it satisfies the equation:

$$\frac{X_{spot}^2}{r_e^2} + \frac{Y_{spot}^2}{r_e^2} + \frac{Z_{spot}^2}{r_p^2} = 1 \qquad \text{(I=15)}$$

or

$$\frac{(X_{sat} + Rd_x)^2}{r_e^2} + \frac{(Y_{sat} + Rd_y)^2}{r_e^2} + \frac{(Z_{sat} + Rd_z)^2}{r_p^2} = 1 \tag{I-16}$$

where r_e is the equatorial radius and r_p is the polar radius of the Earth (The WGS72 values, r_e = 6378.135 km and r_p = 6356.75052 km, are used at NOAA for NOAA satellites). Expanding terms, we have

$$\begin{aligned} &\frac{X_{sat}^2}{r_e^2} + \frac{2X_{sat}Rd_x}{r_e^2} + \frac{R^2d_x^2}{r_e^2} + \frac{Y_{sat}^2}{r_e^2} + \frac{2Y_{sat}Rd_y}{r_e^2} + \frac{R^2d_y^2}{r_e^2} + \\ &\frac{Z_{sat}^2}{r_p^2} + \frac{2Z_{sat}Rd_z}{r_p^2} + \frac{R^2d_z^2}{r_p^2} = 1 \end{aligned} \tag{I-17}$$

or, on simplification,

$$\mathrm{AR}^2 + \mathrm{BR} + \mathrm{C} = 0 \tag{I-18}$$

where

$$A = \frac{d_x^2}{r_e^2} + \frac{d_y^2}{r_e^2} + \frac{d_z^2}{r_p^2} \tag{I-19}$$

$$B = 2\left[\frac{X_{sat}d_x}{r_e^2} + \frac{Y_{sat}d_y}{r_e^2} + \frac{Z_{sat}d_z}{r_p^2}\right] \tag{I-20}$$

$$C = \frac{X_{sat}^2}{r_e^2} + \frac{Y_{sat}^2}{r_e^2} + \frac{Z_{sat}^2}{r_p^2} - 1 \qquad \text{(I-21)}$$

solving for R in equation I-18,

$$R = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \qquad \text{(I-22)}$$

Equation I-18 is a quadratic in R and, if the scan ray does in fact intersect the surface of the Earth (i.e., real and positive roots), B above should always be negative. Since A is always positive and C is positive (whenever the satellite is above the surface of the Earth), then the radical in Eq. I-22 must be less than -B. Therefore, in the case of two different real positive solutions for R, the smaller one, closer to the satellite, is visible to the satellite and the larger one, the point away from the satellite, is on the opposite side of the Earth. As such, the smallest of the two solutions should be taken as the distance of the satellite from the scan spot.
In Equation I-1,

$$\begin{Vmatrix} X_{spot} \\ Y_{spot} \\ Z_{spot} \end{Vmatrix} = \begin{Vmatrix} X_{sat} \\ Y_{sat} \\ Z_{sat} \end{Vmatrix} + R \begin{Vmatrix} d_x \\ d_y \\ d_z \end{Vmatrix} \qquad \text{(I-23)}$$

Figure I.2-4. The earth-centered-inertial and the earth-centered-fixed coordinate systems.

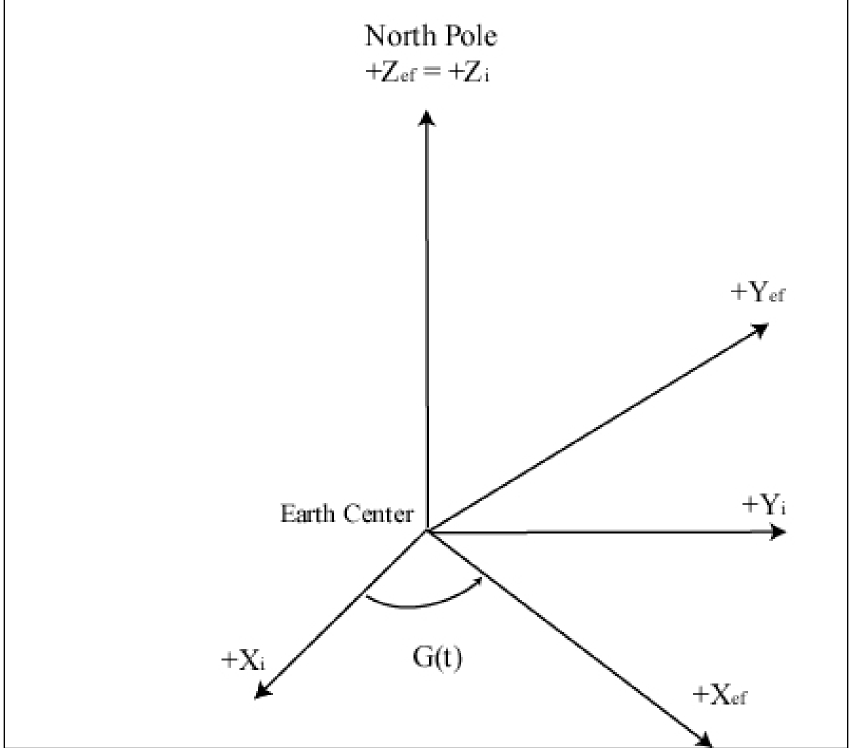

Figure I.2-4. The Earth-centered-inertial Coordinate and the Earth-centered-fixed Coordinate System.

everything on the right hand side is known, so the coordinates of the scan point on the Earth in the inertial coordinate system can be calculated. All that remains is to rotate the Earth using time t obtained from the data sample to calculate a point fixed to the spinning Earth.
The Earth Centered Fixed coordinate system (ECF) rotates with the Earth. It has its center at the center of mass of the Earth with the following defined axes: See Figure I.2-4)

x_{ECF} = the axis from the center of the Earth through Greenwich meridian at the equator (I-24)

y_{ECF} = toward 90 degrees East longitude (I-25)

z_{ECF} = points north along the spin axis of the Earth (I-26)

Next, we calculate the rotation of the ECF system (rotating with the Earth) with respect to the inertially fixed (I) system. The angle G(t), is the rotation of the Greenwich meridian relative to the inertial x-axis. As a function of time,

$$G(t) = G_0 + \dot{G}_1 t_1 + \dot{G}_2 t_2 \tag{I-24}$$

where

$G(t)$ = Hour angle of Greenwich (that is, the eastward angle from the direction of the vernal equinox (0 degrees N, 0 degrees E) measured at the earth's center) at time t (radians)

G_0 = Hour angle of Greenwich at the beginning of the year of interest (radians)

$\dot{G}_1$ = Increase in the hour angle of Greenwich per day (+0.0172027912 radians/day)

t_1 = Day of year for time of interest, t

$\dot{G}_1$ = Rotational rate of the Earth (=6.300388098 radians/day)

t_2 = Fraction of a day for time of interest, t

Values of G_0 can be computed or found in the American Ephemeris and Nautical Almanac for the current year. (These should be updated as appropriate to account for leap seconds.)

In order to transform inertial coordinates to geocentric Earth Centered Fixed coordinates, the following equations are used:

$$X_{ECF} = X_I \cos(G(t)) + Y_I \sin(G(t)) \tag{I-27}$$

$$Y_{ECF} = Y_I \cos(G(t)) - X_I \sin(G(t)) \tag{I-28}$$

$$Z_{ECF} = Z_I \tag{I-29}$$

When the "rotating" coordinates are found, the geocentric latitude, φ_{gc}, and longitude, θ, can then be calculated according to equations I-30 and I-31.

$$\varphi_{gc} = \arctan\left[\frac{Z_{ECF}}{\sqrt{X_{ECF}^2 + Y_{ECF}^2}}\right] \tag{I-30}$$

$$\theta = \arctan\left[\frac{Y_{ECF}}{X_{ECF}}\right] \tag{I-31}$$

Geocentric latitude, φ is the angle between the equatorial plane and a line joining the point, L (the scan spot), on the Earth's surface to the center of mass of the Earth. This is in contrast to the geodetic latitude, φ_{gd}, which is the angle between the normal at L and the plane of the equator. The longitude, θ, is the angle between two meridian planes both containing the earth's axis of rotation; one of the planes contains L, and the other contains the Greenwich meridian.

Values of latitude given on standard maps are usually 'geodetic' latitude. The geodetic latitude, φ_{gd}, of a point on the earth ellipsoid (that is, at Mean Sea Level) has the following value:

$$\varphi_{gd} = \arctan\left[\frac{r_e^2 \cdot Z_{EDF}}{r_p^2 \cdot \sqrt{X_{ECF}^2 + Y_{ECF}^2}}\right] \tag{I-32}$$

(See I.4 "Conversion between Geodetic and Geocentric Latitude")

I.3 DEFINING THE SATELLITE SUBPOINT

There area at least two ways to define the satellite subpoint. The first way is to call it the intersection with the earth ellipsoid surface of a line from the satellite to the earth ellipsoid's center - call this the geocentric subpoint. The second way is to call it the intersection with the earth ellipsoid's surface of a line from the satellite perpendicular to the ellipsoid - call this the geodetic subpoint. (See Figure I.3-1 and imaging that the feature being located is the satellite.) When the satellite is over the North or South Pole or over the Equator, geodetic and geocentric subpoints will be colocated; when it is over 45^0 North or South, the distance between the geocentric and geodetic subpoints will be about 2.5 kilometers for NOAA satellites. Documents describing the NOAA K, L and M Attitude Detection and Control System (ADACS) define the subpoint to be geodetic.

Case 1 - The satellite subpoint is defined to be the geocentric subpoint.

In this case, the unit vector $\vec{P}$ points in the opposite direction from the Satellite position vector. So, if the satellites' position vector in earth-centered-inertial coordinates is

$$\vec{P}_{sat} = (X_{sat}, Y_{sat}, Z_{sat})_I$$

then

$$\hat{P} = (X_p, Y_p, Z_p)_I = \left[\frac{-X_{sat}}{\left|\vec{P}_{sat}\right|}, \frac{-Y_{sat}}{\left|\vec{P}_{sat}\right|}, \frac{-Z_{sat}}{\left|\vec{P}_{sat}\right|}\right]_I$$

where

$$\left|\vec{P}_{sat}\right| = \sqrt{X_{sat}^2 + Y_{sat}^2 + Z_{sat}^2}$$

Case 2 - The satellite subpoint is defined to be the geodetic subpoint.

In this case, the earth-centered-inertial coordinates of the satellite will be known. The problem will be to use them to find the direction cosines from the satellite toward the geodetic subpoint. Again, the position vector of the satellite will be

$$\vec{P}_{sat} = (X_{sat}, Y_{sat}, Z_{sat})_I$$

The magnitude of the component of this vector that is in the equatorial plane is

$$DIST_{sateq} = \sqrt{X_{sat}^2 + Y_{sat}^2}\ .$$

If $DIST_{sateq} = 0$ and $Z_{sat} > 0.0$ or if $DIST_{sateq} = 0$ and $Z_{sat} < 0.0$, or if $Z_{sat} = 0.0$, or if $Z_{sat} = 0.0$ the satellite is over one of the earth's poles or the earth's equator and the geodetic subpoint is the same as the geocentric subpoint.

If $DIST_{sateq} \neq 0$ and $Z_{sat} \neq 0$, then use the equation relating geocentric latitude and geodetic latitude (See "I.4 CONVERSION BETWEEN GEODETIC AND GEOCENTRIC LATITUDE", equation (I-36)).

$$\tan(\varphi_{satgc}) =$$

$$\left[\frac{r_p^2 + h_{sat} \cdot r_p \cdot \sqrt{\left(r_e/r_p\right)^2 \cdot \cos^2(\phi_{satgd}) + \sin^2(\phi_{satgd})}}{r_e^2 + h_{sat} \cdot r_p \cdot \sqrt{\left(r_e/r_p\right)^2 \cdot \cos^2(\phi_{satgd}) + \sin^2(\phi_{satgd})}}\right] \cdot \tan(\phi_{satgd})$$

where h_{sat}, the height of the satellite above the ellipsoid, has been substituted for h_f, the height of the feature of interest, the satellite's geocentric latitude, φ_{satgc}, has been substituted for φ_{gcf}, the geocentric latitude of the feature, and the satellite's geodetic latitude, φ_{satgd}, has been substituted for φ_{gdf}, the geodetic latitude of the feature. The tangent of the satellite's geocentric latitude is given by

$$\tan(\phi_{satgc}) = z_{sat}\ /\ DIST_{sateq}\,.$$

Use it in the above equation, and solve for φ_{satgd} by the procedures given in section, I.4, B "Conversion from Geodetic Latitude", Case 2. The right ascension of the satellite is $\Theta_{sat}=\arctan(Y_{sat}/X_{sat})$. Find the earth-centered-inertial position vector of the geodetic satellite subpoint, (x_s,y_s,z_s), by using equations (I-31), (I-32) and (I-33) and replacing φ_{gdf} with φ_{satgd} and θ_f with Θ_{sat}. The vector from the satellite to its geodetic subpoint will be

$$\vec{P}_{sat2gd} = (x_s - X_{sat}, y_s - Y_{sat}, z_s - Z_{sat})_I$$

and its magnitude will be

$$\left|\vec{P}_{sat2gd}\right| = \sqrt{(x_s - X_{sat})^2 + (y_s - Y_{sat})^2 + (z_s - Z_{sat})^2}\ .$$

The unit vector pointing from the satellite toward its geodetic subpoint will be, then,

$$\hat{P} = (X_p Y_p Z_p)_I = \left[\frac{(x_s - X_{sat})}{\left|\vec{P}_{sat2gd}\right|}, \frac{(y_s - Y_{sat})}{\left|\vec{P}_{sat2gd}\right|}, \frac{(z_s - Z_{sat})}{\left|\vec{P}_{sat2gd}\right|} \right]_I$$

Figure I.3-1. Cross section in the plane of 0^0 E and 180^0 E of the earth ellipsoid showing the geocentric and geodetic latitudes of a feature.

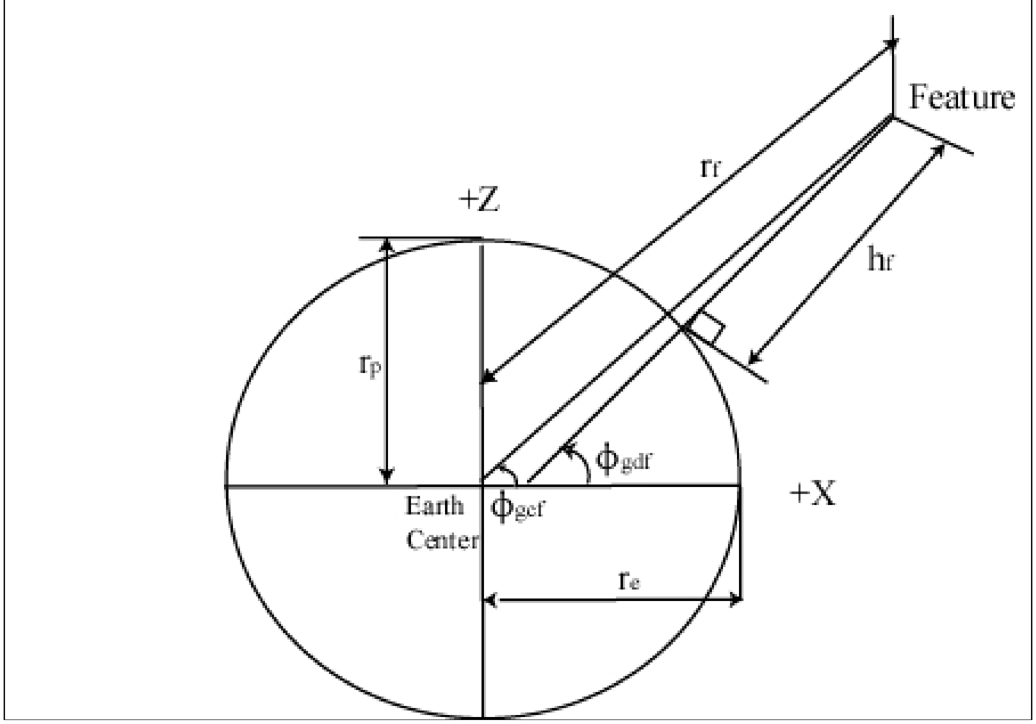

Figure I.3-1. Cross Section in the Plane of 0^0 E and 180^0 E of the Earth Ellipsoid Showing the Geocentric and Geodetic Latitudes of a Feature.

I.4 CONVERSION BETWEEN GEODETIC AND GEOCENTRIC LATITUDE

Figure I.3-1 represents a cross section of the earth with polar radius r_p and equatorial radius r_e. A feature is represented at height h_f above Mean Sea Level. The height is measured along a line through the feature and perpendicular to the earth ellipsoid's surface. The distance of the feature

from the earth's center is r_f.
$\varphi_{gcf} \equiv$ geocentric (or geometric) latitude of the feature.

$-90^0 \leq \varphi_{gcf} \leq 90^0$. North is positive.

$\varphi_{gdf} \equiv$ geodetic latitude of the feature.

$-90^0 \leq \varphi_{gdf} \leq 90^0$. North is positive.

The boundary of the earth's cross section is assumed to be an ellipse formed by the equation

$$\frac{x_s^2}{r_e^2} + \frac{z_s^2}{r_p^2} = 1$$

when the cross section in the xz plane.

A. Conversion from Geodetic to Geocentric Latitude

Suppose that φ_{gdf} is known. Then the slope of the line tangent to the earth's cross section at φ_{gdf} is

$$-\tan(90^0 - \varphi_{gdf}) = 1 / \tan(\varphi_{gdf}).$$

This slope is equal to the derivative dz_s/dx_s. From the equation for the earth's cross section,

$$2 \cdot x_s / r_e^2 + 2 \cdot z_s \cdot (dz_s / dx_s) / r_p^2 = 0$$

$$dz_s / dx_s = -(r_p / r_e)^2 \cdot x_s / z_s$$

So

$$1 / \tan(\varphi_{gdf}) = (r_p / r_s)^2 \cdot x_s / z_s$$

or

$$1 / \tan(\varphi_{gdf}) = (r_p / r_s)^2 \cdot x_s / z_s$$

where (x_s, z_s) is the point on the cross section boundary at geodetic latitude φ_{gdf}
Solving for x_s gives $x_s = (r_e / r_p)^2 \cdot z_s / \tan(\varphi_{gdf})$

Substitution in the equation for the earth’s cross section gives

$$\frac{r_e^2 \cdot z_s^2}{r_p^4 \cdot \tan^2(\varphi_{gdf})} + \frac{z_s^2}{r_p^2} = 1$$

or

$$z_s^2 = \frac{1}{\left[\dfrac{r_e^2}{r_p^4 \cdot \tan^2(\phi_{gdf}) + \left[\dfrac{1}{r_p^2} \right]} \right]} = \frac{r_p^4 \cdot \tan^2(\phi_{gdf})}{r_e^2 + r_p^2 \cdot \tan^2(\phi_{gdf})}$$

$$z_s = \pm \frac{r_p^2 \cdot \tan(\phi_{gdf})}{\sqrt{r_e^2 + r_p^2 \cdot \tan^2(\phi_{gdf})}}$$

$$z_s = \pm \frac{r_p \cdot \tan(\phi_{gdf})}{\sqrt{(r_e / r_p)^2 + \tan^2(\phi_{gdf})}}$$

Substitution off this back in the equation for x_s

$$x_s = \pm \frac{(r_e / r_p)^2 \cdot r_p \cdot \tan(\phi_{gdf})}{\tan(\phi_{gdf}) \cdot \sqrt{(r_e / r_p)^2 + \tan^2(\phi_{gdf})}}$$

These equations for x_s and z_s can be put in the forms

$$x_s = \frac{r_e^2 \cdot \cos(\phi_{gdf}) \cdot \cos(\theta_f)}{r_p \cdot \sqrt{(r_e / r_p)^2 \cdot \cos^2(\phi_{gdf}) + \sin^2(\phi_{gdf})}} \tag{I-31}$$

Where 2_f is the East longitude of the feature,

$$y_s = \frac{r_e^2 \cdot \cos(\phi_{gdf}) \cdot \sin(\theta_f)}{r_p \cdot \sqrt{(r_e / r_p)^2 \cdot \cos^2(\phi_{gdf}) + \sin^2(\phi_{gdf})}}$$

and

$$z_s = \frac{r_p^2 \cdot \sin(\phi_{gdf})}{r_p \cdot \sqrt{(r_e / r_p)^2 \cdot \cos^2(\phi_{gdf}) + \sin^2(\phi_{gdf})}} \tag{I-33}$$

because x_s and y_s will always have the same signs as $\cos(\theta_f)$ and $\sin(\theta_f)$, respectively, and z_s will always have the same sign as $\sin(\varphi_{gdf})$. The coordinates (x_f, y_f, z_f) of the feature can now be calculated.

$$x_f = \left[\frac{r_e^2}{r_p \cdot \sqrt{(r_e / r_p)^2 \cos^2(\phi_{gdf}) + \sin^2(\phi_{gdf})}} + h_f \right] \cdot \cos(\phi_{gdf}) \cdot \cos(\theta_f),$$

$$y_f = \left[\frac{r_e^2}{r_p \cdot \sqrt{(r_e / r_p)^2 \cdot \cos^2(\phi_{gdf}) + \sin^2(\phi_{gdf})}} + h_f \right] \cdot \cos(\phi_{gdf}) \cdot \sin(\theta_f),$$

and

$$z_f = \left[\frac{r_p^2}{r_p \cdot \sqrt{(r_e / r_p)^2 \cdot \cos^2(\phi_{gdf}) + \sin^2(\phi_{gdf})}} + h_f \right] \cdot \sin(\phi_{gdf})$$

Finally, $\tan(\phi_{gcf}) = \dfrac{z_f}{\sqrt{x_f^2 + y_f^2}}$ so

$$\tan(\phi_{gcf}) = \left[\frac{r_p^2 + h_f \cdot r_p \cdot \sqrt{(r_e / r_p)^2 \cdot \cos^2(\phi_{gdf}) + \sin^2(\phi_{gdf})}}{r_e^2 + h_f \cdot r_p \cdot \sqrt{(r_e / r_p)^2 \cdot \cos^2(\phi_{gdf}) + \sin^2(\phi_{gdf})}} \right] \cdot \tan(\phi_{gdf}) \tag{I-34}$$

And the distance to the feature from the earth's center is

$$r_f = \sqrt{x_f^2 + y_f^2 + z_f^2}$$

B. Conversion from Geocentric to Geodetic Latitude

Assume that φ_{gcf}, the geocentric latitude of a feature is known. If the feature's geodetic latitude is to be found, its height above Mean Sea Level must also be known. The feature's height above Mean Sea Level can be calculated if its distance from the center of the earth is known.

Case 1 - h_f the height of the feature above Mean Sea Level, is known

$$\tan(\phi_{gdf}) = \left[\frac{r_e^2 + h_f \cdot r_p \cdot \sqrt{(r_e / r_p)^2 \cdot \cos^2(\phi_{gdf}) + \sin^2(\phi_{gdf})}}{r_p^2 + h_f \cdot r_p \cdot \sqrt{(r_e / r_p)^2 \cdot \cos^2(\phi_{gdf}) + \sin^2(\phi_{gdf})}} \right] \cdot \tan(\phi_{gcf})$$

Since φ_{gdf} will not be known initially for substitution into the right side of the above equation, substitute a guess, starting with $\varphi_{gdf} \approx \varphi_{gcf}$, to compute a better guess. Iterate until the guesses converge.

Case 2 - r_f, the distance of the feature from the center of the earth, is known. Since

$$r_f = \sqrt{x_f^2 + y_f^2 + z_f^2}\,,$$

$$r_f^2 = \left[\frac{r_e^2}{r_p \cdot \sqrt{(r_e / r_p)^2 \cdot \cos^2(\phi_{gdf}) + \sin^2(\phi_{gdf})}} + h_f \right]^2 \cdot \cos^2(\phi_{gdf}) +$$

$$\left[\frac{r_p^2}{r_p \cdot \sqrt{(r_e / r_p)^2 \cdot \cos^2(\phi_{gdf}) + \sin^2(\phi_{gdf})}} + h_f \right]^2 \cdot \sin^2(\phi_{gdf})$$

$$r_f^2 = \frac{r_e^4 \cdot \cos^2(\phi_{gdf})}{r_p^2 \cdot \left[(r_e / r_p)^2 \cdot \cos^2(\phi_{gdf}) + \sin^2(\phi_{gdf}) \right]} +$$

$$\frac{2 \cdot r_e^2 \cdot h_f \cdot \cos^2(\phi_{gdf})}{r_p \cdot \sqrt{(r_e / r_p)^2 \cdot \cos^2(\phi_{gdf}) + \sin^2(\phi_{gdf})}} + h_f^2 \cdot \cos^2(\phi_{gdf}) +$$

$$\frac{r_p^4 \cdot \sin^2(\phi_{gdf})}{r_p^2 \cdot \left[(r_e / r_p)^2 \cdot \cos^2(\phi_{gdf}) + \sin^2(\phi_{gdf}) \right]} +$$

$$\frac{2 \cdot r_p^2 \cdot h_f \cdot \sin^2(\phi_{gdf})}{r_p \cdot \sqrt{(r_e / r_p)^2 \cdot \cos^2(\phi_{gdf}) + \sin^2(\phi_{gdf})}} + h_f^2 \cdot \sin^2(\phi_{gdf})$$

$$r_f^2 = h_f^2 + \frac{2 \cdot h_f \cdot \left[r_e^2 \cdot \cos^2(\phi_{gdf}) + r_p^2 \cdot \sin^2(\phi_{gdf}) \right]}{r_p \cdot \sqrt{(r_e / r_p)^2 \cdot \cos^2(\phi_{gdf}) + \sin^2(\phi_{gdf})}} +$$

$$\frac{r_e^4 \cdot \cos^2(\phi_{gdf}) + r_p^4 \cdot \sin^2(\phi_{gdf})}{r_p^2 \cdot \left[(r_e/r_p)^2 \cdot \cos^2(\phi_{gdf}) + \sin^2(\phi_{gdf}) \right]}$$

$$r_f^2 = h_f^2 + 2 \cdot h_f \cdot r_p \cdot \sqrt{(r_e / r_p)^2 \cdot \cos^2(\phi_{gdf}) + \sin^2(\phi_{gdf})} +$$

$$\frac{r_p^2 \cdot \left[(r_e / r_p)^4 \cdot \cos^2(\phi_{gdf}) + \sin^2(\phi_{gdf}) \right]}{\left[(r_e / r_f)^2 \cdot \cos^2(\phi_{gdf}) + \sin^2(\phi_{gdf}) \right]}$$

Define

$$A \equiv 2 \cdot r_p \cdot \sqrt{(r_e / r_p)^2 \cdot \cos^2(\phi_{gdf}) + \sin^2(\phi_{gdf})}$$

$$B \equiv \frac{r_p^2 \left[(r_e / r_p)^4 \cdot \cos^2(\phi_{gdf}) + \sin^2(\phi_{gdf}) \right]}{\left[(r_e / r_p)^2 \cdot \cos^2(\phi_{gdf}) + \sin^2(\phi_{gdf}) \right]} - r_f^2$$

then $h_f^2 + A \cdot h_f + B = 0$, and this equation could be solved for h_f if φ_{gdf} were known.

$$h_f = \frac{-A \pm \sqrt{A^2 - 4B}}{2}$$

Since φ_{gdf} is not known initially, guess that $\varphi_{gdf} \approx \varphi_{gcf}$, solve for the corresponding h_f, use this as a guess height together with the guess φ_{gdf} is the same as the most recent guess. Then use φ_{gdf} to find a new guess for h_f. Iterate this procedure until both φ_{gdf} and h_f converge to stable values.